Polytope Theory (TCC)

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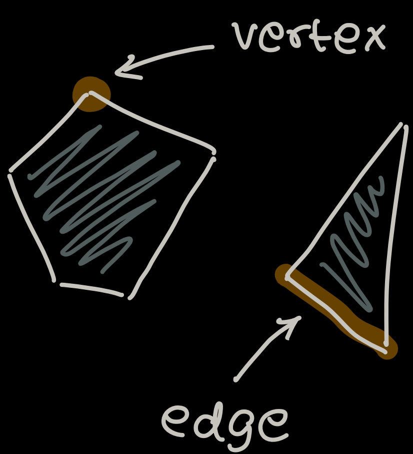
- Monday 10:00 – 12:00
  for 8 weeks until 28th November
- I upload my notes after each lecture
  (but give me some time)
- Lecture will <u>not</u> be recorded
- Feel free to ask questions at any time!
- If you are doing this for a grade, please
  let me know

10/10/2022

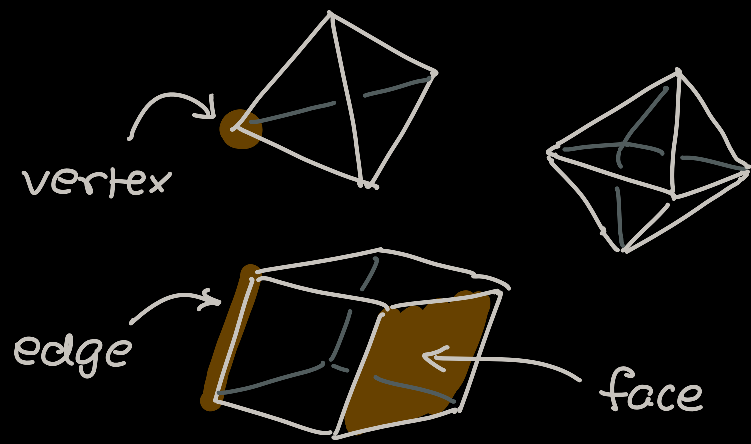
1. Introduction

- What is polytope theory?
- What are polytopes?

2D: polygons



3D: polyhedra



= polygons glued together to form a closed surface

4D, 5D, ..., d-D: ??? $\rightarrow$ polytopes

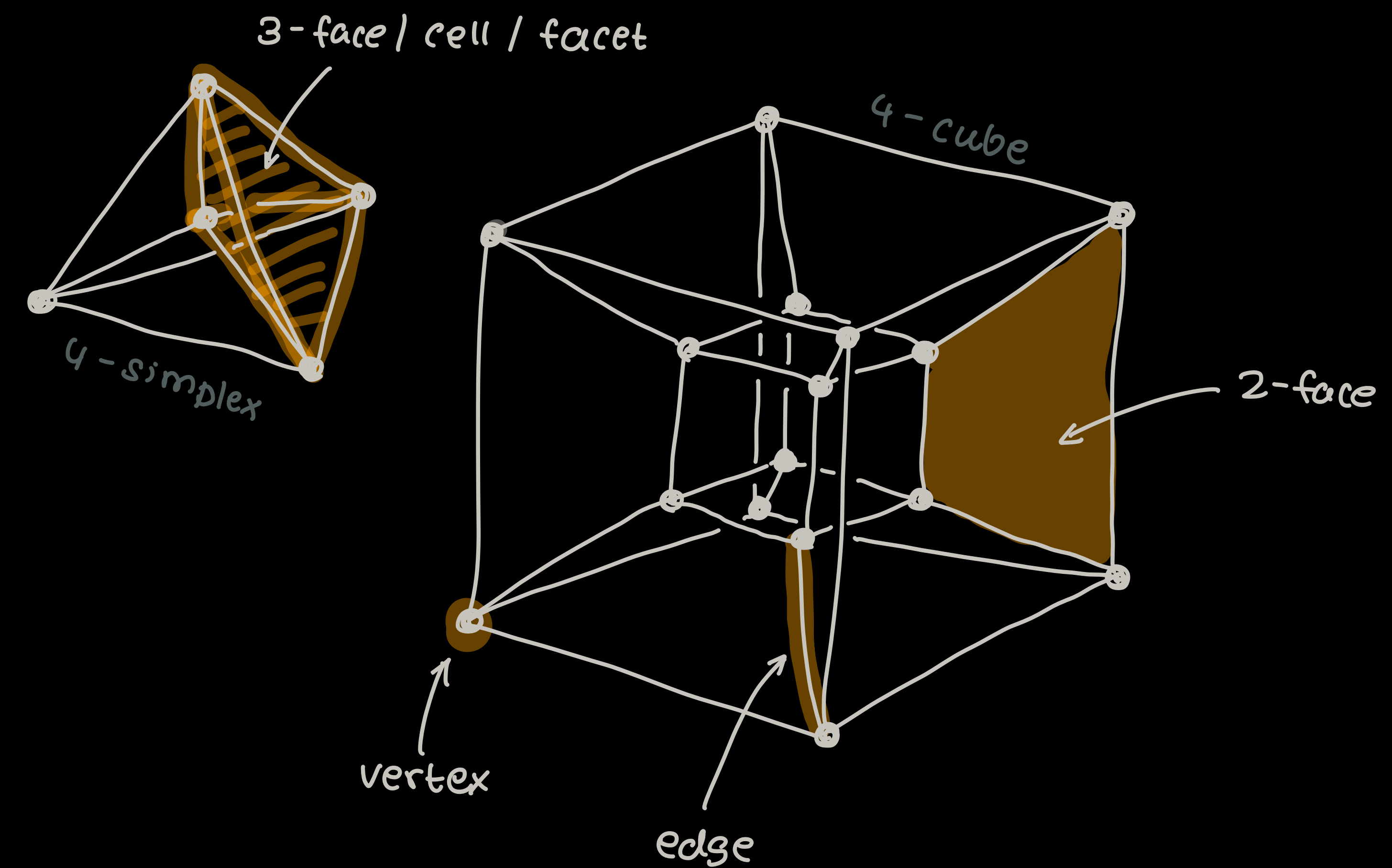
d-dimensional polytope

$\approx$ (d-1)-dim. polytopes glued together to form a closed "surface" in $\mathbb{R}^d$

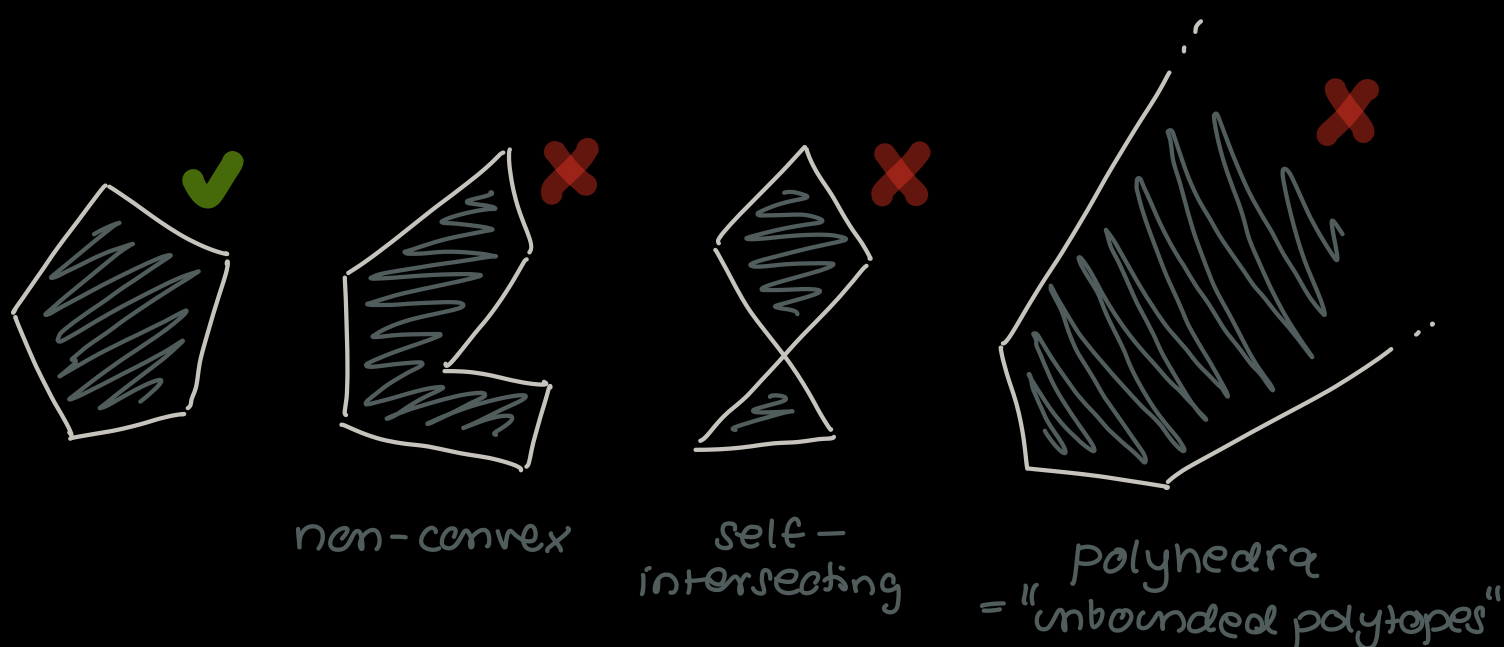
Example: d-dimensional cube $[0,1]^d$

4D:

(Schlegel diagrams)



- We only study convex polytopes



convex := $\forall x, y \in P$: line segment between x and y is in P

Formally defining polytopes

- There are two natural ways to define polytopes

"vertex"

Def: • a V-polytope is the convex hull of finitely many points

convex hull

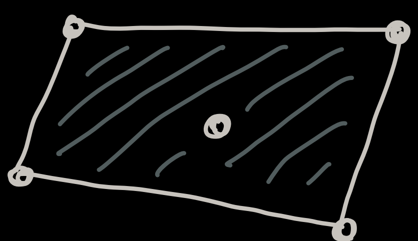
$$P := \operatorname{conv}\{x_1, \dots, x_n\}$$
$$= \Big\{ \sum_i \alpha_i x_i \;\Big|\; \alpha_i \geq 0 \text{ and } \alpha_1 + \dots + \alpha_n = 1 \Big\}$$
$$= \Big\{ \sum_i \alpha_i x_i \;\Big|\; \alpha \in \Delta_n \Big\}$$

convex combination

convex coefficients

$$\Delta_n := \{ \alpha \in \mathbb{R}^n \mid \alpha_i \geq 0 \text{ and } \alpha_1 + \dots + \alpha_n = 1 \}$$

... standard simplex



smallest convex set that contains the points

- a polyhedron is the intersection of finitely many halfspaces.

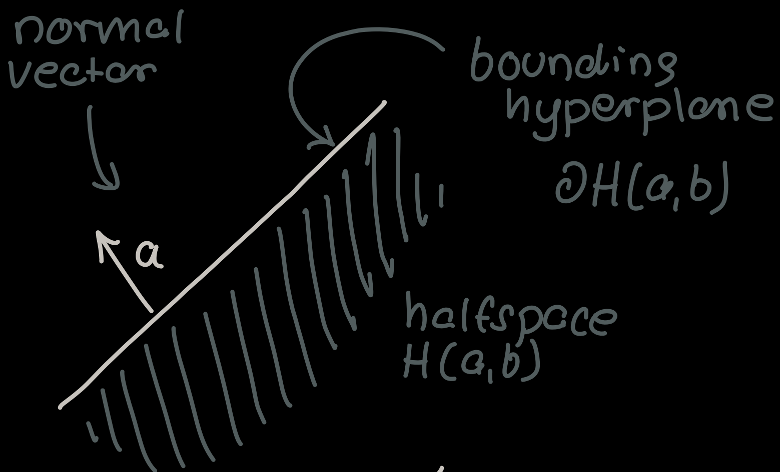
inner product

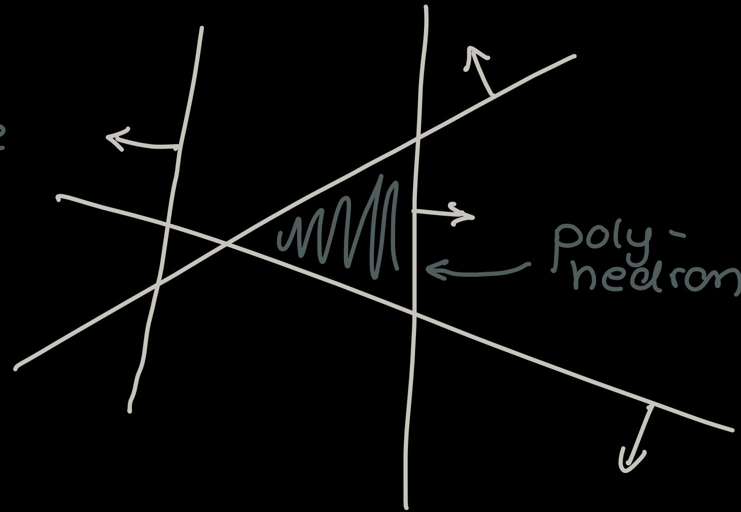
$\{1, \dots, m\}$

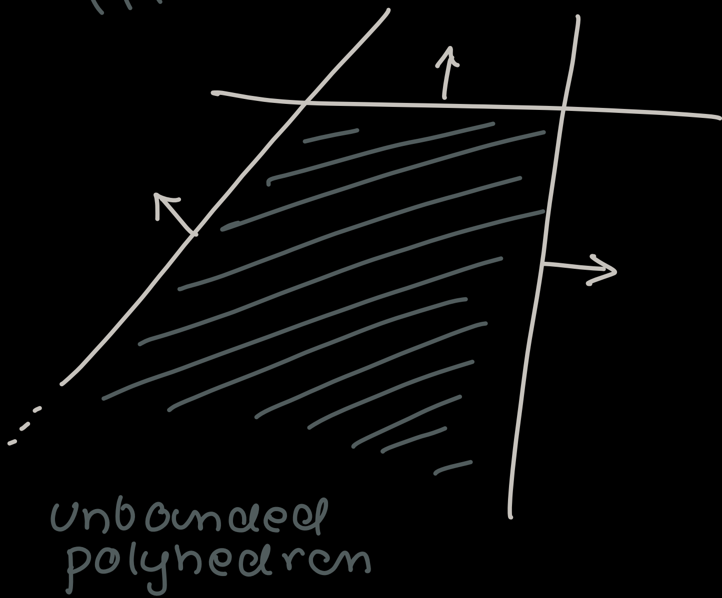
$$P := \{ x \in \mathbb{R}^d \mid \langle a_i, x \rangle \leq b_i \text{ for } i \in [m] \}$$

$\in \mathbb{R}^d \setminus \{0\}$ $\in \mathbb{R}$

$$= \bigcap_i \underbrace{\{ x \in \mathbb{R}^d \mid \langle a_i, x \rangle \leq b_i \}}_{=: H(a_i, b_i) \;\dots\; \text{halfspace}}$$



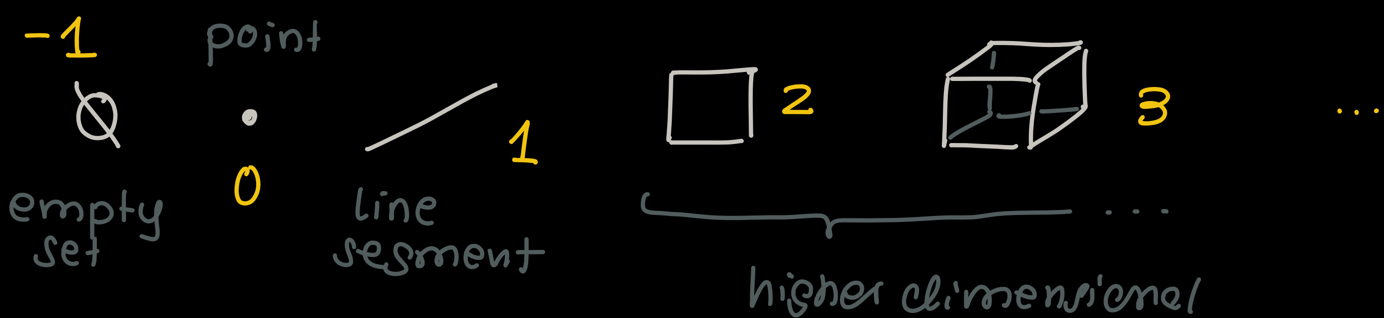




"halfspace"

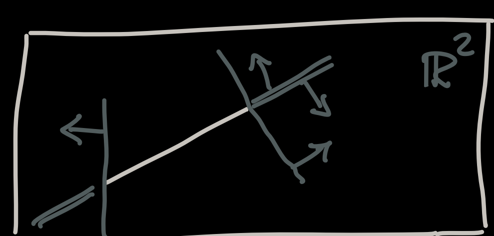
- an $\mathcal{H}$-polytope is a bounded polyhedron

- Later we see: V-polytopes $=$ $\mathcal{H}$-polytopes $=:$ polytopes



- polytopes might <u>not</u> be full-dimensional e.g. line segment in $\mathbb{R}^2$

<u>Ex</u>: write this line segment as an $\mathcal{H}$-polytope



- $\dim(P) := \dim \operatorname{aff}(P)$ ... dimension of P

affine hull

$$\operatorname{aff}(P) := \left\{ \sum_i a_i x_i \;\middle|\; \begin{array}{l} x_1, \dots, x_n \in P \\ a_1 + \dots + a_n = 1 \end{array} \right\}$$

- $\dim(P) = d$

$\longrightarrow$ call it a "d-polytope"

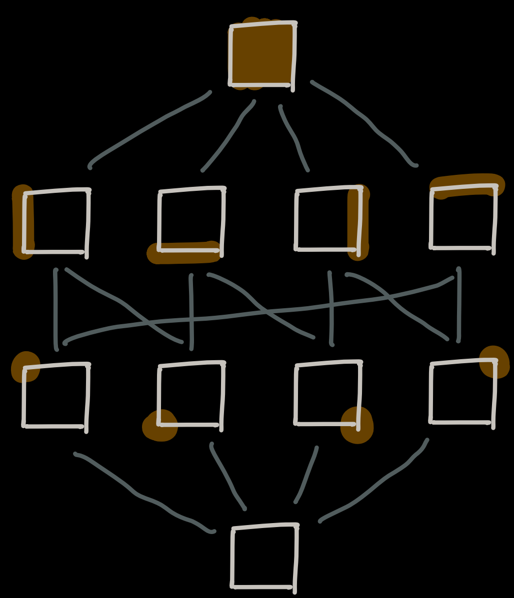
What makes convex polytopes ...

simple / boring ?

- "just" convex bodies
- topological balls ⟶ simply connected, contractible
- up to dimension 3 very well understood

complicated / interesting ?

- additional combinatorial structure



face lattice

- admit a recursive structure
  ⟶ accessible by inductive proofs
- very rich: can approximate every convex body
- counter-intuitive in dimension ≥ 4
- connect geometry and combinatorics

Applications and motivation

1) Linear programming

$$\boxed{\begin{array}{lrcl} \max & 5x+3y-7z & & \\ \text{s.t.} & 3x+3z & \le & 10 \\ & 4y+z & \le & 7 \\ & x,y,z \ge 0 & & \end{array}} \Bigg\} \text{ linear program}$$

- feasible set $:= \left\{ (x,y,z) \;\middle|\; \begin{array}{l} 3x+3z \le 10 \\ 4y+z \le 7 \\ x,y,z \ge 0 \end{array} \right\}$

  is a polyhedron (potentially unbounded)
- understanding their combinatorics helps solving the program (simplex algorithm)

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2) Getting more out of your set of points

$\{x_1, \dots, x_n\} \subset \mathbb{R}^d$

$\longrightarrow$ $\operatorname{conv}\{x_1, \dots, x_n\}$ is a polytope

- we can learn new things about the set of points by studying the combinatorics of this polytope
- E.g. matrix groups, eigenpolytopes, sphere packing

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3) Representing combinatorial objects

- subset $X \subseteq \{1, \dots, n\} \longmapsto$ polytopes

- E.g. (symmetric) edge polytope

  $G = ([n], E)$ standard basis $(0, \dots, \overset{\overset{j}{\downarrow}}{1}, \dots, 0)$

  $$P_G := \operatorname{conv}\{ e_i - e_j \mid ij \in E \} \subseteq \mathbb{1}^{\perp} \subseteq \mathbb{R}^n$$

- E.g. traveling salesperson polytope

  $$P_n := \operatorname{conv}\{ \chi_T \mid T \text{ is a hamiltonian cycle in } K_n \} \subseteq \mathbb{R}^{E(K_n)}$$

  $$(\chi_T)_e := \begin{cases} 1 & \text{if } e \in T \\ 0 & \text{otherwise} \end{cases} \quad \dots \text{ characteristic vector}$$

- E.g. matroid (base) polytopes

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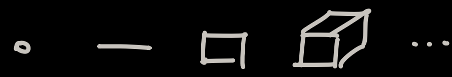
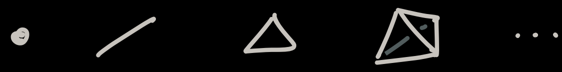
4) Other relations

- crystallography
- hyperbolic geometry
- representation theory
- Hopf algebras
- neural networks (RELU)
- algebraic geometry
  - toric varieties
  - tropical geometry
- geometry of numbers

Examples of polytopes

- polygons: n-gon … n vertices 
- d-dimensional cube : $[0,1]^d$ 

  vertices $= \{0,1\}^d$ **Ex**: find edges, 2-faces etc.

- simplices 
  - the unique d-polytope with $d+1$ vertices
  - all d-simplices are "affinely equivalent"
  - $$\Delta_n := \{\alpha \in \mathbb{R}^n \mid \alpha_i \geq 0 \text{ and } \alpha_1 + \dots + \alpha_n = 1\}$$
    $$= \operatorname{conv}\{e_1, \dots, e_n\} \quad \dots \text{ standard simplex}$$

    $e_i := (0, \dots, 1, \dots, 0)$ … standard basis

    Note: $\Delta_n \subset \frac{1}{n}\mathbb{1} + \mathbb{1}^\perp$ orthogonal complement, $(1, \dots, 1)$

    $\longrightarrow$ Δ_n is an $(n-1)$-polytope

  - every polytope is the projection of some simplex

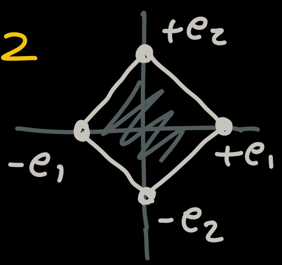
    $$P = \operatorname{conv}\{x_1, \dots, x_n\} = \left\{\sum_i \alpha_i x_i \mid \alpha \in \Delta_n\right\} \subset \mathbb{R}^d$$
    $$= \{T\alpha \mid \alpha \in \Delta_n\} = T\Delta_n$$

    $$T = \begin{bmatrix} | & & | \\ x_1 & \cdots & x_n \\ | & & | \end{bmatrix} \in \mathbb{R}^{d \times n} \;\hat{=}\; \text{projection}$$

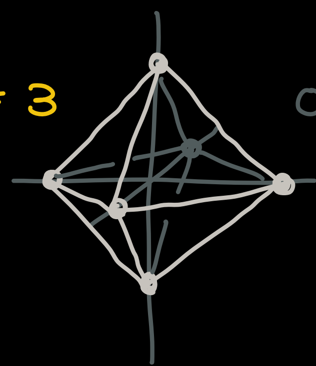
- crosspolytopes (= orthoplexes)

$$P := \operatorname{conv}\{\pm e_i\}$$

d = 2



d = 3



octahedron

- the five Platonic solids

artifacts of 3D

3-simplex	3-cube	3-crosspolytope		
tetrahedron	cube	octahedron	icosahedron	dodecahedron
4	6	8	20	12

- higher-dimensional regular polytopes

4D:
- 4-simplex
- 4-cube
- 4-crosspolytope

} exist in all dimensions

- 24-cell
- 120-cell
- 600-cell

} artifacts of 4D

Typical questions asked in polytope theory

- How many different polytopes are there?

→ when are two polytopes "the same"

→ how to enumerate different types?

→ for 3-polytopes (Wormald – Bender, 1988)

$$\sim \frac{1}{2^2 3^5 nm(n+m)} \binom{2m}{n+3} \binom{2n}{m+3} \qquad \left(\begin{array}{l} n \text{ vertices} \\ m \text{ faces} \end{array}\right)$$

- In how many ways can a combinatorial type be realized?

→ "space of realizations"?

- How does a typical polytope look like? (random polytopes)
- How to reconstruct a polytope from combinatorial data and some geometric data? Is this even possible?

- How many faces can a polytope have?
  What relations exist between the face numbers?

  $$V - E + F = 2$$ (Euler's polyhedral formula)

- Computing with polytopes:
  - How to convert V- to H-polytopes and back?
  - How to enumerate faces?
  - How to compute volumes? (#P hard)

2. Basics

2.1 Minkowski-Weyl Theorem

= "main theorem of polytope theory"

= V-polytopes = $\mathcal{H}$-polytopes

we only show this direction for now

Thm: If $P \subset \mathbb{R}^d$ is an $\mathcal{H}$-polytope then it is a V-polytope

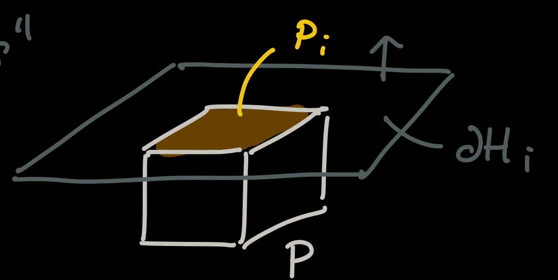
Proof:

$\{\langle a_i, x\rangle \le b_i\}$

- $P = \bigcap \mathcal{H}$ with $\mathcal{H} = \{H(a_i, b_i) \mid i \in [m]\}$ of minimal size
- we can assume that P is full-dimensional, as otherwis consider it as subset of $\mathrm{aff}(P)$

  Ex: P is an $\mathcal{H}$-poly. in $\mathbb{R}^d \Longleftrightarrow P$ is an $\mathcal{H}$-poly in $\mathrm{aff}(P)$
- we proceed by induction on dimension d
- induction base: $d \in \{-1, 0, 1\} \longrightarrow$ obviously V-polytopes
- induction step:

  Idea: collect vertices from the "faces"



  - $P_i := P \cap \partial H(a_i, b_i)$
  - P_i is an $\mathcal{H}$-polytope:

  $$P_i = \bigcap_k H(a_k, b_k) \cap \overline{H}(a_i, b_i)$$

  opposite halfspace

  $\overline{H} := \mathrm{closure}(\mathbb{R}^d \setminus H)$

  - P_i is non-emply and of dimension $< d$
  - by induction hypothesis: $P_i = \mathrm{conv}\, V_i$
  - set $V := \bigcup_i V_i = \{x_1, \dots, x_n\}$

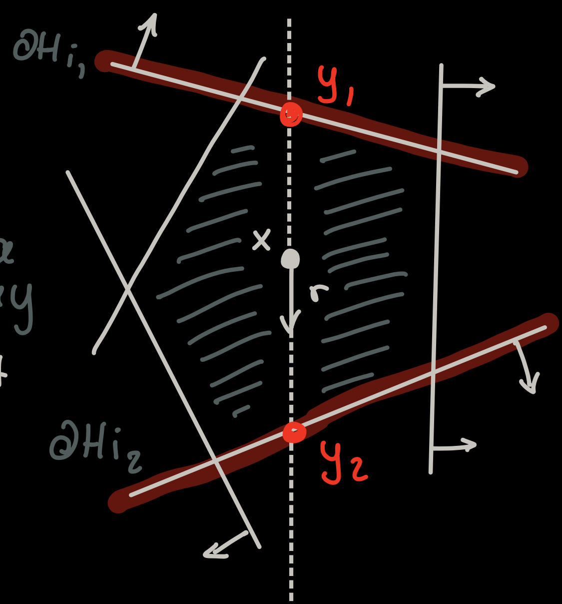
Claim: $P = \operatorname{conv} V$

- $\operatorname{conv} V \subseteq P$:

  for some k

  - $x_i \in P$ because $x_i \in V_k \subseteq P_k \subseteq P$
  - since $V \subseteq P$ and P convex $\longrightarrow$ $\operatorname{conv} V \subseteq P$

- $P \subseteq \operatorname{conv} V$:



  - fix an arbitrary point $x \in P$
  - choose some $r \in \mathbb{R}^d$ Idea: cast a ray
  - Since P is bounded, there are $t_1 \le 0$ and $t_2 \ge 0$ so that

$$\left.\begin{aligned} y_1 &:= x + t_1 r \\ y_2 &:= x + t_2 r \end{aligned}\right\} \in \partial P$$

  - by continuity $y_k \in \partial H(a_{i_k}, b_{i_k}) \cap P = P_{i_k}$

    for some i_k

$$= \operatorname{conv} V_{i_k} \subseteq \operatorname{conv} V$$

$$\left.\begin{aligned} &\bullet\ y_k \in \operatorname{conv} V \\ &\bullet\ x \in \operatorname{conv}\{y_1, y_2\} \end{aligned}\right\} \quad \underline{\text{Ex}}:\ x \in \operatorname{conv} V$$

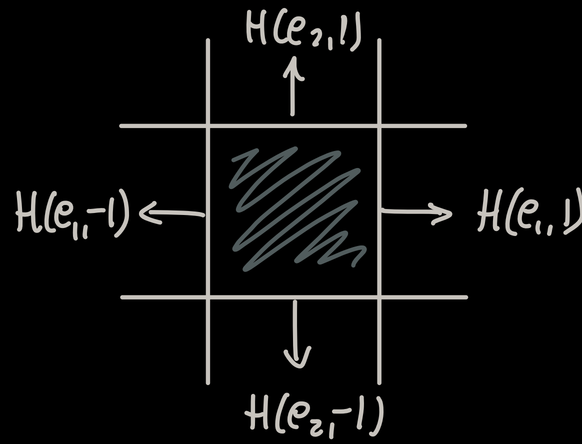
$\square$

- We have shown that all polytopes are V-polytopes
- How to do this algorithmically?

  $\longrightarrow$ Fourier-Motzkin elimination (see Ziegler's book)
- Can this be efficient? Not really :(

  e.g. Consider converting a d-cube from $\mathcal{H}$-poly to V-poly

$$[-1,1]^d = \bigcap_i H(e_i, \pm 1)$$



→ you need $2d$ halfspaces
BUT 2^d points in every V-representation

$$[-1,1]^d = \operatorname{conv}\{0,1\}^d$$

→ A polynomial time conversion algorithm cannot exist